MATH 122
MATHEMATICAL INDUCTION

Let P_n be a statement involving the positive integer n.

Examples:

a. $n^2 = n \times n$
b. $2n$ is an even number
c. $2n + 1$ is an odd number
d. $\frac{n+1}{n} = 1$
e. n is divisible by 3
f. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Which of the above statements are true for ALL positive integral values of n?

Answer: a, b, c and f (f is not obviously true)
Note that d is not true for any value of n, while e is only true for
$n = 3, 6, 9, \ldots$

f is true for the following reason:

$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$ = sum of the first n terms of this arithmetic series (S_n).

$= n+1$

$= n+1$

$= n+1$

Thus $S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(1 + n) = \frac{n(n+1)}{2}$

Notation:

If P_n is a statement,
Then P_1 is this statement where n = 1
P_2 is this statement where n = 2, etc.

Example:

Let P_n be: $n^2 + n$ is positive.
Then P_1 says, "$1^2 + 1$ is positive",
P_4 says, "$4^2 + 4$ is positive", etc.

AXIOM OF MATHEMATICAL INDUCTION

Let P_n be a statement (i.e.P_1, P_2, P_3, ... etc. are defined).

Furthermore, let the following 2 conditions exist:

1. P_1 is true
2. P_{k+1} is true whenever P_k is true. (k is any positive integer.)

Conclusion: P_n is true for $n = 1, 2, 3, \ldots$

i.e. P_1, P_2, P_3, P_4,... are <u>all</u> true.

EXAMPLE 1:

PROVE $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ by mathematical induction. (i.e. Prove that P_n is true for n = 1, 2, 3, ...)

P_n: $1+2+3+\cdots+n=\frac{n(n+1)}{2}$

IS P_1 TRUE?

P_1: $1=\frac{1(1+1)}{2}$, i.e., 1 = 1 SO P_1 is true.

ASSUME P_k is true: i.e., $1+2+3+\cdots+k=\frac{k(k+1)}{2}$

PROVE P_{k+1} is true: i.e., Prove $1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$

USUALLY YOU START ALL WITH THE LEFT SIDE OF P_{k+1}:

$$1+2+3+\cdots+(k+1)=$$

$$\underbrace{1+2+3+\cdots+k}_{\text{left side of } P_k}+(k+1)=$$

$$\Updownarrow$$

$$\frac{\overbrace{k(k+1)}^{\text{right side of } P_k}}{2}+(k+1)=$$

$$\frac{k(k+1)+2(k+1)}{2}=\text{right side of } P_{k+1}$$

Therefore, P_n true for n = 1, 2, 3,...

EXAMPLE 2

SHOW THAT: P_n: 5^n - 1 is divisible by 4.

IS P_1 TRUE?

P_1: $\frac{5^1-1}{4}=\frac{5-1}{4}=\frac{4}{4}=1$ Therefore: P_1 is true.

ASSUME P_k: 5^k-1 divisible by 4.

or $\frac{5^k-1}{4}=q$, is an integer

Then $5^k-1=4q$ and $5^k=4q+1$

PROVE P_{k+1}: 5^{k+1} - 1 divisible by 4.

$$5^{k+1}-1=$$

$$5^k\cdot 5-1=$$

$$\overbrace{(4q+1)}^{\text{replaces } 5^k}\cdot 5-1=4q\cdot 5+1\cdot 5-1$$

$$=\underbrace{4q\cdot 5}_{\text{divisible by 4}}+\underbrace{5-1}_{\text{divisible by 4}}$$

Therefore P_n TRUE for n = 1, 2, 3,...

MATH 122
MATHEMATICAL INDUCTION PROBLEMS

Use induction to prove that each of the following formulas is true for each positive integer *n*.

1. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
2. $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + (2n-1)(2n) = \frac{n(n+1)(4n-1)}{3}$
3. $\frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \cdots + \frac{n}{2} = \frac{n(n+1)}{4}$
4. $2 + 6 + 10 + \cdots (4n-2) = 2n^2$
5. $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$
6. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$
7. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \cdots + n(n+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5)$

By induction show that:

8. $3^n - 1$ is divisible by 2.
9. $5^n - 1$ is divisible by 4.
10. $7^n - 1$ is divisible by 6.
11. $8^{2n} - 1$ is divisible by 63.
12. $6^{2n} - 1$ is divisible by 35.
13. $9^{2n} - 1$ is divisible by 80.
14. $n^2 - 3n + 4$ is even.
15. $2n^3 - 3n^2 + n$ is divisible by 6.
16.
    a. Show: If $2 + 4 + 6 + ... + 2n = n(n+1) + 2$ is true for $n = j$, then it is true for $n = j + 1$.
    b. Is the formula true for all *n*?