MATH254 Final Exam Practice Problems

Your in-class final exam will consist entirely of problems similar to the following, with numbers and functions changed but worded the same way. It is assumed you will thoroughly master all the problems on this handout and errors <u>in understanding</u> on your in-class final will earn less partial credit than on previous quizzes.

Differential Families

x^n	$\{1, x, x^2, \ldots, x^n$
e^{ax}	$\{e^{ax}\}$
sin bx or cos bx	{ sin bx , cos bx}
$x^n e^{ax}$	$\{ e^{ax}, x e^{ax}, x^2 e^{ax}, \ldots x^n e^{ax}\}$
e^{ax} sin bx or e^{ax} cos bx	{ e^{ax} sin bx, e^{ax} cos bx}
x^n sin ax or x^n cos ax	{cos ax, sin ax, x cos ax, x sin ax, x^2 cos ax, x^2 sin ax, …, x^n cos ax, x^n sin ax}

<u>Cauchy-Euler equations</u>

Method 1) $am^2 + (b - a)m + c = 0$

Two distinct roots: $y = c_1 x^{m_1} + c_2 x^{m_2}$

Repeated roots: $y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$

Complex roots: $y = x^a \left(c_1 \cos(b \ln x) + c_2 \sin(b \ln x)\right)$

Method 2) $x = e^t, \quad \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

Variation of Parameters

$y_p = u_1 y_1 + u_2 y_2 \qquad u_1 = \int \frac{W_1}{W}, \; u_2 = \int \frac{W_2}{W}$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W(y_1, y_2)}, \quad where \;\; W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

For L-R-C series circuits, $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

Also, $\mathcal{L}\{1\} = \frac{1}{s}$ $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as} F(s)$

$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$ $\mathcal{L}\{\mathcal{U}(t-a)\} = e^{-as}\frac{1}{s}$

$\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$ $\mathcal{L}\{f(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

1. Find the general solution (in simplified form) of each of the following.

a) $(xy - x)\,dx = (xy^2 + x - y^2 - 1)\,dy$ b) $\left(1 + \frac{y}{x^2 + y^2}\right)dx - \frac{x}{x^2 + y^2}dy = 0$

c) $dx = (x + y^2)\,dy$ d) $y' - y = xy^{½}$

e) $x^2 \frac{dy}{dx} + y^2 = xy$ f) $\sin(x^2)\frac{dy}{dx} = xy\cos(x^2)$

2. Whenever a solution $y_1(x)$ is known for an equation $y'' + P(x)\,y' + Q(x)\,y = 0$, we can try to find another solution $y_2(x)$ of the form $u(x)\,y_1(x)$, and this leads to the formula

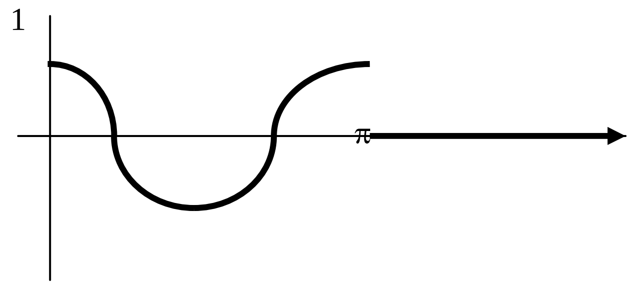
$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}dx}{[y_1(x)]^2}.$$

Use this formula to find a 2nd solution $y_2(x)$ for each of the following. Verify that $y_1(x)$ and $y_2(x)$ are linearly independent (use the Wronskian) and find the general solution of each of the following.

a) $x(x+1)\,y'' + (2 - x^2)\,y' - (2 + x)\,y = (x+1)^2$, $x > 0$; $y_1(x) = \frac{1}{x}$

b) $(1 - 2x - x^2)\,y'' + 2(1+x)\,y' - 2y = 0$; $y_1(x) = x + 1$

3. Solve $y'' + 4y = f(t)$, $y(0) = 0$, $y'(0) = 1$, where $f(t) = \begin{cases} \cos 2t & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$



The function f(t) can be interpreted as an impressed voltage that acts on an electrical circuit for only a short period of time and then is turned off.

4. Find the general solution (simplified, of course) of the following equations.

a) $y^{(4)} - 4y^{(3)} + 7y'' - 6y' + 2y = 0$

b) $y''' + y'' + 3y' - 5y = 0$

c) $y''' - 2y'' - 5y' + 6y = e^x + x^2$

5. A large tank contains 81 gallons of brine (salt water) in which 20 lbs of salt are dissolved. Brine containing 3 lbs of dissolved salt per gallon pours into the tank at the rate of 5 gallons per minutes. The mixture, kept uniform by stirring, pours out of the tank at the rate of 2 gallons per minute. Find a general equation for the number of lbs of salt in the tank at elapsed time t minutes. In particular, how much salt (to the nearest tenth of a pound) is in the tank when 39 minutes have elapsed? Show all details clearly.

6. Newton's Law of Cooling states that the rate of temperature decrease of an object immersed in a medium of constant temperature T_m is proportional to the difference $T - T_m$, where T is the object's temperature. If a just-cooked pie is removed from a 350° F oven to cool in a room with temperature 72° F, and 5 minutes later the temperature of the pie is 200° F, find the equation for the temperature T of the pie at elapsed time t minutes. Also compute (to the nearest tenth) how many minutes it will take for the pie to cool to 100° F (and be ready to eat).

7. Find the general solution of each of the following.

a) $x^2y'' - 4xy' + 6y = 2x^4 + x^2$

b) $x^2y'' - xy' + y = x^3$

8. Find the general solution in power series in x for each of the following. Find at least the first 3 nonzero terms (if there are that many) of each particular solution. Show clearly the recurrence relation(s) used.

a) $y'' - 4xy' - 4y = e^{x}$

b) $2x^{2} y'' + xy' - (x + 1) y = 0$

9. Solve the given differential equation.

$$y' + y = f(t),\ y(0) = 0, \text{ where } f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & t \ge 1 \end{cases}$$

10. Solve the following system.

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = -5x$$

$$\frac{dx}{dt} + \frac{dy}{dt} = -x + 4y$$

11. Given the differential equation: $xy'' + y' = 0$ (on the interval $(0, \infty)$).

a. Is $y_1 = 1$ a solution?
b. Is $y_2 = \ln x$ a solution?
c. Is $y_3 = 3$ a solution?
d. Is $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ the general solution? Why or why not?

12. Questions with short answers.

a. It is a fact that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $y\,y'' + (y')^2 = 0,\ x > 0$ but that $y = c_1 + c_2\sqrt{x}$ is, in general, not a solution. Why doesn't this contradict any of our theorems?

b. Suppose y_p is a solution of $y'' + p(x)y' + q(x)y = g(x)$ with $g(x) \neq 0$ and y_c a solution of the corresponding homogeneous equation. Then:

i) $y_c + y_p$ is a solution of $y'' + p(x)y' + q(x)y = g(x)$ True □ False □

ii) $y_c - y_p$ is a solution of $y'' + p(x)y' + q(x)y = g(x)$ True □ False □

c. A general solution of $y'' + y = x$ is given by $y = c_1 \cos x + c_2 \sin x + c_3 x$ where c_1, c_2 and c_3 are arbitrary constants. True □ False □

13. Suppose you decide to save some money for retirement. You initially deposit \$1000 into an account paying 5¼ % compounded continuously. You then deposit \$50 a month for 30 years. How much will be in the account at that time?

14. Determine a particular solution for: $y''' + y' = \tan x,\ 0 < x < \pi/2$

15. Find a general solution for the homogeneous linear differential equation with constant coefficients whose auxiliary equation is:

a. $(m + 5)^2 (m - 2)^3 (m^2 + 1)^2 = 0$

b. $m^4 (m - 1)^2 (m^2 + 2m + 4)^2 = 0$

16. Find the homogeneous linear differential equation of smallest order with constant coefficients whose general solution contains the following terms:

a) x^2, sin 2x.

b) x sin 3x

17. Solve: $x^3 y''' - 2x^2 y'' + 3xy' - 3y = x^2,\ x > 0$

18. Solve each of the following initial value problems.

a. $y'' - 2y' + 5y = -8e^{-t},\ y(0) = 2,\ y'(0) = 12$

b. $x'' + tx = 0,\ x(0) = 1,\ x'(0) = 0$

c. $y'' - 2y' + y = \dfrac{e^x}{x},\ y(1) = 0,\ y'(1) = 1$

d. $x^2 y'' - 4xy' + 4y = 0,\ y(1) = -2,\ y'(1) = -11$

ANSWERS

1. a. separable, $\frac{y^2}{2} + y + 2\ln(y-1) = x + \ln(x-1) + C$

   b. exact, $x + \tan^{-1}\left(\frac{x}{y}\right) = C$ *or* $y = x\cot(C - x)$

   c. linear in x, use $\mu(y) = e^{\int (-1)\,dy} = e^{-y}$ to obtain $x = -y^2 - 2y - 2 + ce^y$

   d. Bernoulli with n = ½ , $y^{1/2} = -x - 2 + ce^{x/2}$

   e. Bernoulli, $\frac{1}{y} = \frac{\ln x + c}{x}$ *or* $y = \frac{x}{\ln x + c}$ *or* $e^{x/y} = cx$

   f. separable, $y = c\sqrt{\sin(x^2)}$

2. a. $y = c_1x^{-1} + c_2e^x - \frac{1}{2}x - 1$

   b. $y = c_1(x + 1) + c_2(x^2 + x + 2)$

3. $y = \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - \frac{1}{4}(t-\pi)\sin 2(t-\pi)U(t-\pi)$

4. a. $y = c_1e^x + c_2xe^x + e^x(c_3\cos x + c_4\sin x)$

   b. $y = c_1e^x + e^{-x}(c_2\cos 2x + c_3\sin 2x)$

   c. $y = c_1e^x + c_2e^{3x} + c_3e^{-2x} - \frac{1}{6}xe^x + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{37}{108}$

5. $A = 3(81+3t)^{5/2} - \frac{4174.7282}{(81+3t)^{2/3}}$, 471.1 lbs

6. $T = 72 + 278e^{-.1551t}$, 14.8 minutes

7. Both Cauchy-Euler

   a. $y = c_1x^2 + c_2x^3 + x^4 - x^2\ln x$

   b. $y = c_1x + c_2x\ln x + \frac{1}{4}x^3$

8. a. $y = c_0(1 + 2x^2 + 2x^4 + \ldots) + c_2(x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \ldots) + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{13}{24}x^4 + \ldots$

b. $y = c_0\left(x + \frac{1}{5}x^2 + \frac{1}{70}x^3 + \frac{1}{1890}x^4 + ... \right) + c_1\left(x^{-1/2} - x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{18}x^{5/2} - ... \right)$

9. $y = 1 - e^{-t} - 2U(t-1) + 2e^{-(t-1)}U(t-1)$

10. $x = c_1e^{5t} + c_2\cos 2t + c_3 \sin 2t$, $y = -6c_1e^{5t} + ½ c_3\cos 2t - ½ c_2\sin 2t$

11. a. yes
b. yes
c. yes
d. no, the three functions are linearly dependent ($y_3 = 3y_1$). Since the equation is a 2nd order equation, you only need two linearly independent solutions (y_2 with either of the other two gives a fundamental set).

12. a. DE isn't linear
b. True, False
c. False (y_p doesn't contain arbitrary constants)

13. $A = -11428.57 + 12428.57e^{.0525t}$, \$48610.64

14. $y_p = \ln(\sec x) - \sin x (\ln (\sec x + \tan x))$ (any constants , i.e. the one, get 'absorbed' into the constant of y_c.

15. a. $e^{-5x}(c_1 + c_2x) + e^{2x}(c_3 + c_4x + c_5x^2) + (\cos x)(c_6 + c_7x) + (\sin x)(c_8 + c_9x)$

b. $c_1 + c_2x + c_3x^2 + c_4x^3 + e^x(c_5 + c_6x) + (e^{-x}\cos\sqrt{3}x)(c_7 + c_8x) + (e^{-x}\sin\sqrt{3}x)(c_9 + c_{10}x)$

16. a) $y^{(5)} + 4y''' = 0$ b) $y^{(4)} + 18y'' + 81y = 0$

17. $y = c_1x + c_2x \ln x + c_3x^3 - x^2$

18. a. $y(t) = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}$

b. $x = 1 - \frac{t^3}{6} + \frac{t^6}{180} + ...$

c. $y = \frac{e-1}{e}e^x + \frac{1-e}{e}xe^x + xe^x \ln|x| = (1-x)(e-1)e^{x-1} + xe^x \ln|x|$

d. $y = x - 3x^4$